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Quantifying Correlation Uncertainty Risk in Credit Derivatives Pricing [†]

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Abstract: We propose a simple but practical methodology for the quantification of correlation risk in the context of credit derivatives pricing and credit valuation adjustment (CVA), where the correlation between rates and credit is often uncertain or unmodelled. We take the rates model to be Hull–White (normal) and the credit model to be Black–Karasinski (lognormal). We summarise recent work furnishing highly accurate analytic pricing formulae for credit default swaps (CDS) including with defaultable Libor flows, extending this to the situation where they are capped and/or floored. We also consider the pricing of contingent CDS with an interest rate swap underlying. We derive therefrom explicit expressions showing how the dependence of model prices on the uncertain parameter(s) can be captured in analytic formulae that are readily amenable to computation without recourse to Monte Carlo or lattice-based computation. In so doing, we crucially take into account the impact on model calibration of the uncertain (or unmodelled) parameters.

Keywords: perturbation expansion; Green’s function; model risk; model uncertainty; credit derivatives; CVA; correlation risk

1. Introduction

1.1. Model Risk Management

Much effort is currently being invested into managing the risk faced by financial institutions as a consequence of model uncertainty. One strand to this effort is an increased level of regulatory scrutiny of the performance of the model validation function, both in terms of ensuring that adequate testing is performed for all models used for pricing and risk management purposes and of enforcing a governance policy that only models so tested are so used. As is stated in the Supervision and Regulation Letter of [US Federal Reserve \(2011\)](#):

An integral part of model development is testing, in which the various components of a model and its overall functioning are evaluated to show the model is performing as intended; to demonstrate that it is accurate, robust, and stable; and to evaluate its limitations and assumptions.

Another concern is model risk monitoring and management. Here, the idea is that, having validated models and examined the associated uncertainty, the risk department should monitor and report on the risk faced by a financial institution, ideally so that senior management can, based on “risk appetite”, make informed decisions about model usage policy. According to [US Federal Reserve \(2011\)](#):

Validation activities should continue on an ongoing basis after a model goes into use to track known model limitations and to identify any new ones. Validation is an important check during periods

of benign economic and financial conditions, when estimates of risk and potential loss can become overly optimistic and the data at hand may not fully reflect more stressed conditions...Generally, senior management should ensure that appropriate mitigating steps are taken in light of identified model limitations, which can include adjustments to model output, restrictions on model use, reliance on other models or approaches, or other compensating controls.

Here, the notion of best practice is less well established, in particular because different institutions adopt different approaches to measuring and reporting model risk. In what probably remains the most definitive book¹ on the subject, [Morini \(2011\)](#) asserts:

You will see that not even among quants there is consensus about what model risk is.

Indeed, there are currently regular industry events held at which practitioners and managers from financial institutions share and discuss their views on current best practice and how this should evolve. It is not therefore possible to enforce specific regulatory standards in this area, although regulators do take an interest in how banks perform the model risk governance function.

Central to the task of monitoring and managing model risk or uncertainty is the challenge of how to measure it. Current practice tends to be a mix of qualitative and quantitative metrics. While the former are easier to implement, the latter are preferable in terms of the level of control that can be exercised, particularly if the model risk can be quantified in monetary terms. However, the fact that no commonly agreed methodology has emerged and such methodologies as have been proposed tend not to lend themselves to implementation by practitioners means that it is not easy to make progress in this area.

A common approach taken by financial institutions has been to consider the reserves taken by the finance function to account for model parameter and/or calibration uncertainty as a proxy measure of model risk. This is arguably less than satisfactory for a number of reasons, not least that the purpose of reserves is to provide a protective buffer against, rather than a precise measure of, model risk.

The present paper represents a proposed compromise between rigour and practicality to furnish model risk metrics against which risk appetite can be compared. We consider specifically rates–credit correlation risk in relation to credit derivatives pricing, but it is suggested the methodology may be applicable more widely to other types of model risk and a wider class of financial instruments.

1.2. Layout of the Paper

We begin in Section 2 by reviewing previous methodologies that have been proposed for the quantification of model risk, before formally outlining our own proposed approach. We go on in Section 3 to describe the model we shall consider for pricing credit derivatives in the context of stochastic interest rates and credit intensity, indicating how, following [Turfus \(2017a\)](#), a Green's function solution can be obtained as a perturbation expansion, under the assumption that both of these rates are small. In Section 4, the key results of [Turfus \(2017a\)](#) obtained by application of this Green's function to CDS pricing are summarised.

In Section 5, similar expressions for the PV of other credit derivatives, specifically credit-contingent interest rate swaps (including with capped or floored Libor) and contingent CDS with an interest rate swap underlying are used in conjunction with those developed in Section 2 to assess the level of model risk associated with the uncertain parameter(s). Finally, in Section 6, we present some concluding remarks and a number of directions for possible future work.

¹ It should be mentioned that the author in his book eschews the idea that any one book should aim to be definitive.

2. Model Risk Methodology

2.1. Previous Work

A number of authors have previously visited the question of what constitutes an appropriate methodology for the quantification of model risk in pricing financial derivatives. In his pioneering work on the subject, [Cont \(2004\)](#) proposes two approaches. In the first, a family of plausible models is envisaged, each calibrated to all relevant market instruments and then used to price a given portfolio of exotic derivatives. The degree of variation in the prices that are observed provides a measure of the intrinsic uncertainty associated with modelling the price of the portfolio. A second approach, taking account of the fact that not all models are amenable to calibration to market instruments, compares the models by penalising them for the pricing error associated with calibration instruments. The pricing errors for multiple instruments can be combined using various choices of norm, giving rise to a number of possible measures of model risk.

While intuitively attractive, neither of these approaches appears to have been adopted by practitioners. This is likely a consequence of the cost of implementing multiple models and re-pricing under them. Financial institutions usually only have very few models implemented, often just one, capable of pricing a given exotic option. Furthermore, regulatory pressure has recently been towards standardising pricing of financial derivatives by restricting or even reducing the size of the set of available models, which mitigates against the adoption of the kind of approach envisaged by [Cont \(2004\)](#).

More recently, [Glasserman and Xu \(2014\)](#) have proposed an alternative approach based on maximising the model error subject to a constraint on the level of plausibility. The approach starts from a baseline model and finds the worst-case error that would be incurred through a deviation from the baseline model, given a precise constraint on the plausibility of the deviation. Using relative entropy to constrain model distance leads to an explicit characterization of worst-case model errors. In this way, they are able to calculate upper bounds on model error. They show how their approach can be applied to the problems of portfolio risk measurement, credit risk, delta hedging and counterparty risk measured through credit valuation adjustment (CVA).

Although this approach has the attraction of a rigorous definition and, according to the authors, is amenable to convenient Monte Carlo implementation, it has the disadvantage that an entropy constraint specified a priori is not the sort of concept that risk managers are likely to be comfortable with in defining or expressing risk appetite. However, it is central to the whole approach. Furthermore, the approach has the disadvantage that it probably offers too much laxity in allowing the joint probability distribution function governing risk factors to vary freely subject only to the entropy constraint. Many of the perturbed distributions, including those giving rise to worst-case errors, would likely be deemed “unrealistic” by practitioners for reasons that cannot easily be encoded through entropy considerations. An approach that allows the user to be more specific about what is believed to be “known” and with what degree of certainty using a parametrisation more closely related to market variables would probably be preferred.

For example, the consensus among practitioners might be that the “best” interest rate model would be somewhere between a normal and a lognormal process. However, under the proposal of [Glasserman and Xu \(2014\)](#), if a Hull–White (normal) model were chosen as the baseline, deviations towards lognormal and away from it would be penalised equally. However, we are really only interested in assessing the impact of the former.

In his review of the subject [Morini \(2011\)](#), while lamenting the paucity of the model risk literature, comes down against excessive use of mathematical formalism and numerics that can serve to obscure the all-important link between specific modelling assumptions and the variability of prices that can emerge therefrom and advocates a middle path between that and “formal compliance or simple techniques to produce numbers that are acceptable to put in reports, but lacks [sic] the quantitative approach that would be needed to understand models deeply.” A useful insight into the sorts of techniques that are currently used to put numbers into reports has recently been provided by

Joshi (2017). A more forward-looking perspective on the strategies that are being developed and implemented for model risk management in financial institutions is provided by Crespo et al. (2017).

In the present work, we look to build to some extent on the basic philosophy of Cont (2004), but simplifying the methodology in compliance with the advocacy of Morini (2011) and avoiding the prohibitive cost of implementing multiple numerical models. We shall suggest that key to making progress is the ability to assess, at least to a good approximation, the impact of more advanced model features without necessarily having to implement them explicitly in a fully working model.

To this end, we propose that asymptotic analysis that has, in the author's view, been under-used in risk management offers a fruitful way forward, certainly in the context of credit derivatives pricing with which we shall mainly be concerned here. To this end, we draw extensively on the asymptotic analysis of the impact of rates–credit correlation on CDS pricing in Turfus (2017a) and on contingent CDS in Turfus (2017b). A key advantage here is that the output of the model risk assessment process is in the form of analytic formulae rather than numerical routines. The relative transparency of the former will often furnish insight about which model configurations and inputs give rise to the greatest degree of uncertainty in terms of output prices. The alternative approach that is typically followed in model risk analysis is to sample the phase space of all possible market conditions, model parameterisations and product characteristics in a generally rather unsystematic way and infer conclusions from such data as are gleaned in the process. An obvious drawback of this approach is that there are no guarantees that the worst cases where the potential discrepancies are largest are uncovered; or how representative the numbers in the dataset are of the phase space that has been “sampled”.

We suggest that a further reason why the pricing model risk methodologies that have been proposed in the literature appear to have had limited traction in financial institutions lies in the structure of those institutions. Specifically, it is an issue that authority to make judgements in relation to which models are used, how they are configured and calibrated and how model risk is assessed is inevitably devolved across multiple functions: front office, market risk, model validation, finance, CVA desk, etc. For that reason, we propose that a simplified approach, whereby questions about how price-relevant model parameters are assigned and how the uncertainty associated with these is assessed, can be considered separately from issues around how their values and assumed uncertainty levels (or distributions) impact on prices. Our focus in this paper will by choice be on the latter. We note in this context that a further practical advantage of our proposed approach over the more mathematically sophisticated approach of Glasserman and Xu (2014) is that the intuition of practitioners tasked with the responsibility to assess parameter uncertainty levels is better attuned to more immediate parameters like correlation than the more abstract concept of entropy.

2.2. Proposed Framework

We formally state the problem we are looking to address as follows. Consider a model $\mathcal{M}(s, \rho)$ which we wish to use as the basis for pricing a portfolio Φ containing derivatives D_k , $k = 1, 2, \dots, m$. Here, $s = (s_1, s_2, \dots, s_n)$, with s_i the value of a credit spread (suitably defined) associated with maturity T_i , $i = 1, 2, \dots, n$, and ρ the correlation between rates and credit, the appropriate value of which is unknown and furthermore *not* readily ascertainable from market data, or else not in practice modelled. We wish to consider and indeed quantify, to a good approximation, the dependence of the portfolio price on the correlation parameter.

To this end, we consider the calibration at time $t = 0$ of credit spreads in the model to a vector of market prices $p = (p_1, p_2, \dots, p_n)$ for calibration instruments $\{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n\}$. Let us express the result of such calibration for an assumed value of ρ formally as

$$s = f(\rho; p) \quad (1)$$

for some $f : [-1, 1] \rightarrow \mathbb{R}^n$. Since the market instrument prices are considered fixed for the time $t = 0$ of interest, we shall for convenience generally omit the explicit dependence on p_i in the following,

in particular writing component-wise simply $s_i = f_i(\rho)$. There may be other market instruments to which the model is calibrated, but, if the generated prices of these are not sensitive to ρ (or only very weakly so), we need not consider them explicitly in our analysis here.

Let us then denote the price calculated for derivative D using model $\mathcal{M}(s, \rho)$, thus calibrated by $V(D; \mathcal{M}(s, \rho))$. Let us further introduce the shorthand notation

$$V_k(\rho) = V(D_k; \mathcal{M}(f(\rho), \rho)). \quad (2)$$

From the asymptotic analyses that will be described below, we see that the dependence of credit derivatives prices on ρ tends to be well captured as a linear function thereof.² An appropriate measure of the model risk associated with pricing the derivative portfolio with an assumed value $\rho = \rho_0$ is on this basis obtained by use of the linear approximation:

$$R(\Phi; \rho_0, \Delta\rho) := \Delta\rho \sum_{k=1}^m \frac{\partial V_k(\rho)}{\partial \rho} \Big|_{\rho=\rho_0} \quad (3)$$

with $\Delta\rho$ an estimate of the level of uncertainty or inaccuracy associated with the representation of the parameter ρ . Performing the required differentiation on (2), we see

$$\frac{\partial V_k(\rho)}{\partial \rho} \Big|_{\rho=\rho_0} = \left(\sum_{i=1}^n \frac{\partial V(D_k; \mathcal{M}(s; \rho))}{\partial s_i} f'_i(\rho) + \frac{\partial V(D_k; \mathcal{M}(s; \rho))}{\partial \rho} \right) \Big|_{s=f(\rho), \rho=\rho_0}. \quad (4)$$

We seek a convenient practical means of determining $f'_i(\rho_0)$. To that end, we note that the i th calibration condition can be expressed as

$$V(\mathcal{I}_i; \mathcal{M}(s; \rho)) = p_i, \quad (5)$$

leading by the same token to

$$\left(\sum_{j=1}^i \frac{\partial V(\mathcal{I}_i; \mathcal{M}(s; \rho))}{\partial s_j} f'_j(\rho_0) + \frac{\partial V(\mathcal{I}_i; \mathcal{M}(s; \rho))}{\partial \rho} \right) \Big|_{s=f(\rho), \rho=\rho_0} = 0. \quad (6)$$

Here, we assume that the model is bootstrapped by applying the calibration conditions in order of maturity, whence there will be no dependence of the price of \mathcal{I}_i on s_j for $j > i$. The partial derivatives in this equality can be computed conveniently by use of the asymptotic modelling approach described below. From this, the values of $f'_i(\rho_0)$ can be inferred recursively. Substituting in (4) and then (3) gives rise to our representation of the model risk. On the basis of our assumption of approximate linear dependence of prices on ρ over the range of interest, we propose that our evaluations of $f'_j(\rho_0)$, with typically $\rho_0 = 0$, can consistently be used in place of $f'_j(\rho)$.

Our suggestion here is that, if we can derive analytic approximations to instrument prices taking into account the uncertain model parameters, this opens the way to obtaining analytic representations of the partial derivatives in (4) and so to obtaining an estimate of the model risk more conveniently and in a more transparent form than otherwise. We will illustrate our approach with some examples from the credit derivatives area, with which the author is most familiar.

² For other choices of model parameter than ρ , this will often still tend to be the case on the basis that the uncertain/unmodelled parameter will be of secondary importance; if this were not so, the consequent high degree of uncertainty introduced into pricing would compromise the utility of the pricing algorithm.

3. Two-Factor Asymptotic Model

3.1. Underlying Processes

Our modelling approach will be to represent the interest rate r_t and the credit default intensity λ_t (of a named debt issuer) as correlated mean-reverting short rate processes. In this respect, our approach is similar to that pioneered by [Schönbucher \(1999\)](#) who took both processes to be normal mean-reverting diffusions, in other words governed by the Gaussian short rate model of [Hull and White \(1990\)](#). Solutions were in his case found by constructing a two-dimensional tree. As was pointed out by [Schönbucher \(1999\)](#), it is a straightforward matter to extend his model to non-Gaussian processes.

A number of authors have followed this suggestion taking the credit process to be lognormal, governed by a [Black and Karasinski \(1991\)](#) short rate model, which, although less tractable than a Gaussian model, ensures that credit spreads stay positive (and thus that survival probabilities are decreasing functions of time). [Jobst and Zenios \(2001\)](#) sought to price portfolios of bonds, modelling the credit spread for securities in a given rating class in this way, coupled with a Hull–White interest rate model, but also allowing rating class migrations to take place. A similar approach with only rates and credit default risk was used by [Cortina \(2007\)](#) to provide analytic solutions for the prices of defaultable bonds in the assumed absence of correlation, and by [Pan and Singleton \(2007\)](#) who considered the joint distribution of credit spreads and default loss rates implied by CDS market data.

We will follow the latter authors in taking the interest rate process to be normal, as proposed by [Hull and White \(1990\)](#), and the credit intensity process to be lognormal, so ensuring positive intensities, following [Black and Karasinski \(1991\)](#). The correlation ρ between these two processes we take to be the uncertain model parameter of interest, although we could equally extend our framework to allow consideration of the credit mean reversion rate, or even its volatility as uncertain model parameters. We shall find it convenient to work with auxiliary variables x_t and y_t satisfying the following Ornstein–Uhlenbeck processes:

$$dx_t = -\alpha_r x_t dt + \sigma_r(t) dW_t^1, \quad (7)$$

$$dy_t = -\alpha_\lambda y_t dt + \sigma_\lambda(t) dW_t^2, \quad (8)$$

where $\alpha_r, \alpha_\lambda \in \mathbb{R}^+$, $\sigma_r, \sigma_\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are piecewise continuous functions and dW_t^1, dW_t^2 are correlated Brownian motions under the risk-neutral measure for $t \geq 0$ with

$$\text{corr}(W_t^1, W_t^2) = \rho. \quad (9)$$

These auxiliary variables are related to the interest short rate r_t and the credit default intensity λ_t , respectively, by

$$r_t = \bar{r}(t) + r^*(t) + x_t, \quad (10)$$

$$\lambda_t = (\bar{\lambda}(t) + \lambda^*(t))\mathcal{E}(y_t). \quad (11)$$

Note that here and below we use subscript- t to indicate stochastic processes; otherwise, all functions of t are assumed to be deterministic. Here, $\bar{r}(t)$ is the instantaneous forward rate (which we allow to take on negative values), $\bar{\lambda}(t)$ the associated credit spread (see (15) below) and $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation of a process X_t . Here, all functions and processes in (10) map $[0, T_m] \rightarrow \mathbb{R}$ (although in practice $r^*(t)$ will be strictly positive for $t > 0$), while those in (11) map $[0, T_m] \rightarrow \mathbb{R}^+$, where $t = 0$ is the “as of” date and T_m the longest maturity date for which the model is calibrated.

The interest rate model obtained in this way is of Hull–White type and the credit intensity model Black–Karasinski. It is well known that both of these models can be made risk-neutral by specification of suitable respective drifts, which determine in turn the configurable functions $r^*(\cdot)$ and

$\lambda^*(\cdot)$ uniquely. The required form of $r^*(t)$ and $\lambda^*(t)$ is in practice obtained by calibration of the model to ensure satisfaction of the no-arbitrage conditions set out below, assumed to apply for all $t \in (0, T_m]$. As we shall see, it proves possible on this basis to determine these functions uniquely in asymptotic form under the conditions specified above.

3.2. The No-Arbitrage Conditions

The formal no-arbitrage constraints that determine the functions $r^*(t)$ and $\lambda^*(t)$ are as follows:

$$E \left[e^{-\int_0^t r_s ds} \right] = D(0, t), \quad (12)$$

$$E \left[e^{-\int_0^t (r_s + \lambda_s) ds} \right] = B(0, t), \quad (13)$$

under the martingale measure for $0 < t \leq T_m$, where

$$D(t_1, t_2) = e^{-\int_{t_1}^{t_2} \bar{r}(s) ds} \quad (14)$$

is the t_1 -forward price of the t_2 -maturity zero coupon bond and

$$B(t_1, t_2) = e^{-\int_{t_1}^{t_2} (\bar{r}(s) + \bar{\lambda}(s)) ds} \quad (15)$$

the corresponding risky bond price. We shall assume the bond prices can be ascertained at the initial time $t = 0$ from the market, whence we can view (14) and (15) as defining the forward rate $\bar{r}(t)$ and associated credit spread $\bar{\lambda}(t)$, respectively.

3.3. Derivation of Governing PDE

We consider the general problem of pricing a cash security with maturity T whose payoff depends on x_T . We will also look below at protection instruments whose payoff may depend on τ and x_τ , where τ is a stopping time in $(0, T)$. We introduce the convenient shorthand notation that, for a process X_t and function $f: \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$\mathcal{E}_x(f(t)X_t) := \mathcal{E}(f(t)X_t)|_{X_t=x}, \quad (16)$$

in terms of which we can re-write (10) and (11) as $r_t = r(x_t, t)$ and $\lambda_t = \lambda(y_t, t)$, where

$$r(x, t) := \bar{r}(t) + r^*(t) + x_t, \quad (17)$$

$$\lambda(y, t) := (\bar{\lambda}(t) + \lambda^*(t))\mathcal{E}_y(y_t). \quad (18)$$

Writing the price of the security at time $t \in [0, T \wedge \tau]$ as $f_t^T = f(x_t, y_t, t)$, we can infer by application of the converse of the Feynman–Kac theorem to (7) and (8) in the standard manner that the function $f(x, y, t)$ satisfies the following backward diffusion equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - r(x, t) - \lambda(y, t) \right) f(x, y, t) = 0, \quad (19)$$

where

$$\mathcal{L} := -\alpha_r x \frac{\partial}{\partial x} - \alpha_\lambda y \frac{\partial}{\partial y} + \frac{1}{2} \left(\sigma_r^2(t) \frac{\partial^2}{\partial x^2} + 2\rho\sigma_r(t)\sigma_\lambda(t) \frac{\partial^2}{\partial x \partial y} + \sigma_\lambda^2(t) \frac{\partial^2}{\partial y^2} \right) \quad (20)$$

with in general $\lim_{t \rightarrow T^-} f_t^T = P(x_T)$ for some payoff function $P(x)$.³ In the absence of closed form solutions to (19) and guided by the work of Hagan et al. (2005), Pagliarani and Pascucci (2011) and Horvath et al. (2017), we propose a perturbation expansion approach as follows.

For both short rate models, we apply a ‘low rates’ assumption. To this end, we define small parameters

$$\epsilon_r := \frac{1}{\alpha_r T_m} \int_0^{T_m} |\bar{r}(t)| dt, \quad (21)$$

$$\epsilon_\lambda := \frac{1}{\alpha_\lambda T_m} \int_0^{T_m} \bar{\lambda}(t) dt. \quad (22)$$

We assume that $\bar{r}(t)$ and $\sigma_r(t)$ are $\mathcal{O}(\epsilon_r)$, while $\bar{\lambda}(t)$ is $\mathcal{O}(\epsilon_\lambda)$. The scaling of $r^*(t)$ and $\lambda^*(t)$ is inferred as part of the calculation. We presage our conclusions by writing

$$r^*(t) \sim \gamma_{2,0}^*(t), \quad (23)$$

$$\lambda^*(t) \sim \gamma_{1,1}^*(t) + \gamma_{0,2}^*(t), \quad (24)$$

with $\gamma_{ij}^*(t) = \mathcal{O}(\epsilon_r^i \epsilon_\lambda^j)$. We rewrite (19) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \bar{r}(t) - \bar{\lambda}(t) - \phi_\epsilon(x, y, t) \right) f(x, y, t) = 0, \quad (25)$$

where

$$\phi_\epsilon(x, y, t) := h(x, t) + g(y, t), \quad (26)$$

$$h(x, t) := r(x, t) - \bar{r}(t), \quad (27)$$

$$g(y, t) := \lambda(y, t) - \bar{\lambda}(t). \quad (28)$$

We take advantage of the assumed smallness of $\phi_\epsilon(\cdot)$ to seek a Green’s function solution $G(x, y, t; \xi, \eta, v)$ for (25) as a joint power series in ϵ_r and ϵ_λ , asymptotically valid in the limit as these two parameters tend to zero. The details of this calculation are provided in Turfus (2017a) with the conclusions summarised in Appendix A below.

4. CDS Pricing

We next consider how we can use our Green’s function to price a credit default swap (CDS) analytically under an assumed rates–credit correlation, again following closely Turfus (2017a). Although this is a vanilla instrument, its use in calibration means that it is nonetheless important to have analytic formulae.

4.1. Fixed Coupon Leg

If, as proposed, the risky discount factors $B(t_1, t_2)$ are assumed known, a coupon payment made for a payment period $[t_{i-1}, t_i]$ with coupon c and $t_i > 0$ can be straightforwardly priced as

$$PV_{Coupon}^{(i)} = cB(0, t_i)\Delta_i, \quad (29)$$

with Δ_i the relevant year fraction.

³ We shall henceforth write for simplicity $f(x, y, T)$ as a shorthand for $\lim_{t \rightarrow T^-} f(x, y, t)$.

4.2. Protection Leg

Turfus (2017a) shows how the price of a protection leg can be derived by solving a nonhomogeneous version of (25) with the forcing function $-(1-R)\lambda(y, t)$ on the r.h.s., where R is the assumed recovery level of the referenced debt. The result obtained is, in the present notation,

$$\begin{aligned} PV_{Prot} &\sim (1-R) \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda(\eta, v) G(0, 0, 0; \xi, \eta, v) d\xi d\eta dv \\ &\sim (1-R) \int_0^T B(0, v) (\bar{\lambda}(v) + \Delta\lambda(v)) dv \end{aligned} \quad (30)$$

per unit notional with $\mathcal{O}(\epsilon_r(\epsilon_r + \epsilon_\lambda))$ relative error, where

$$\Delta\lambda(v) := \gamma_{1,1}^*(v) - \bar{\lambda}(v) \int_0^v e^{-\alpha_r(v-u)} I_{r\lambda}(u) du, \quad (31)$$

with $\gamma_{1,1}^*(\cdot)$ given by (A28) and $I_{r\lambda}(\cdot)$ by (A9), provides an $\mathcal{O}(\epsilon_r\epsilon_\lambda)$ adjustment to the leading order result. Here, the first term in the expression for $\Delta\lambda(\cdot)$ comes from the application of $G_{0,0}$ to $\lambda^*(\cdot)$ and the second from the application of $G_{1,0}$ to $\bar{\lambda}(\cdot)$. Note that, in the absence of correlation, $\Delta\lambda(\cdot) = 0$ and the value of protection is as given under the assumption of deterministic rates.

4.3. Calibration to CDS Market

If we consider our model to be calibrated to risky bond prices, the calibration is at this stage completely specified, at least to second order accuracy. In particular, we see that $f'_i(\rho) = 0$ in (4), simplifying our task.

Alternatively if, as is often the case, the calibration is to a term structure of CDS rates, we can take the market prices p_i associated with maturities T_i to be zero and the associated market instruments to be ATM CDS. Let us further suppose that the function $\bar{\lambda}(t)$ can be taken as piecewise constant between the T_i , given say by

$$\bar{\lambda}(t) = \lambda_i, \quad t \in (T_{i-1}, T_i]. \quad (32)$$

with $T_0 \equiv 0$. We can then take the s_i introduced in Section 2.2 above to be given by

$$s_i T_i = \sum_{j=1}^i \lambda_j (T_j - T_{j-1}). \quad (33)$$

Inference of the $f'_i(\rho)$ is then straightforward, but our task is simplified if we are willing to consider only the leading order impact of calibration, whence we can neglect the $\mathcal{O}(\epsilon_\lambda^2)$ indirect impact of the λ_i through the (risky) discount factors in favour of their $\mathcal{O}(\epsilon_\lambda)$ direct impact in the context of default-driven payoffs. A straightforward calculation gives rise to the conclusion that

$$f'_i(\rho) \approx -\frac{T_i - T_{i-1}}{\rho T_i} \frac{\int_0^{T_i} B(0, u) \Delta\lambda(u) du}{\int_{T_{i-1}}^{T_i} B(0, u) du} \quad (34)$$

with expected $\mathcal{O}(\epsilon_\lambda)$ relative errors, which is consistent with our use of (30).⁴ Equipped with this additional information, we are in a position to assess the model uncertainty associated with other derivative types priceable by our model.

⁴ The errors can in addition be expected to approximate to near zero since the calibration swaps are assumed to be at the money, whence the (risky) discounting affects both legs almost equally.

5. Calculating Correlation Risk

5.1. Interest Rate Swap Extinguisher

An interest rate swap extinguisher is an interest rate swap where the cash flows are contingent on survival of a named debt issuer. We have already considered credit-contingent fixed flows in Section 4.1 above. We now look to price credit-contingent Libor flows. The payoff at time t_i for a payment period $[t_{i-1}, t_i]$ is given in the notation defined in Appendix A below by

$$\begin{aligned} \text{Payoff} &= X^{t_i}(x_{t_{i-1}}, t_{i-1})^{-1} - 1 \\ &\sim D(t_{i-1}, t_i)^{-1}(1 + x_{t_{i-1}} B^*(t_i - t_{i-1})) - 1, \end{aligned} \quad (35)$$

with errors $= \mathcal{O}(\epsilon_r^2)$. The calculation for the PV of this Libor flow contingent on no default was performed by Turfus (2017a). This was found to be given by

$$PV_{\text{Libor}}^{(i)} \sim B(0, t_i) \left(\frac{1 - \Delta L^{(i)}}{D(t_{i-1}, t_i)} - 1 \right), \quad (36)$$

with $\mathcal{O}(\epsilon_r^2 + \epsilon_\lambda^2)$ relative error, where

$$\Delta L^{(i)} := \int_0^{t_i} \bar{\lambda}(v) \phi_A^{(i)}(v) dv, \quad (37)$$

$$\phi_A^{(i)}(v) := \int_{t_{i-1}}^{t_i} \gamma(u, v) I_{r\lambda}(u \wedge v) du, \quad (38)$$

$$\gamma(u, v) := \begin{cases} e^{-\alpha_\lambda(v-u)}, & u \leq v, \\ e^{-\alpha_r(u-v)}, & u > v \end{cases} \quad (39)$$

and $I_{r\lambda}(\cdot)$ is given by (A9). We here use the binary operators \wedge and \vee to represent min and max respectively. In conclusion, the fair price of a payer extinguisher will be

$$PV_{\text{Extinguisher}} = \sum_{i=1}^N \left(PV_{\text{Libor}}^{(i)} - PV_{\text{Coupon}}^{(i)} \right) \quad (40)$$

assuming the payments are synchronised. (The extension of the calculation if payments are not synchronised is trivial.)

A comparison of analytic calculations based on (40) against the results of a finite difference solution of the underlying PDE is reproduced from Turfus (2017a) in Figure 1 to illustrate a typical parameter dependence structure and to indicate the level of accuracy that is furnished by our asymptotic method. The swap extinguisher paid quarterly Libor + 100 bp spread and received a quarterly 400 bp fixed coupon against a swap notional of 100. The credit default intensity was taken to be 770 bp, with a local vol of 60% and a mean reversion rate of 0.3. The 10-year swap rate was taken to be 80 bp with a short rate local volatility increasing from 20 bp to 70 bp and a mean reversion rate of 0.25. As can be seen from the graph, the use of our linear approximation approach to the model risk is a good one, with the discrepancy between the two modelling approaches in all cases less than 0.1 bp of notional.

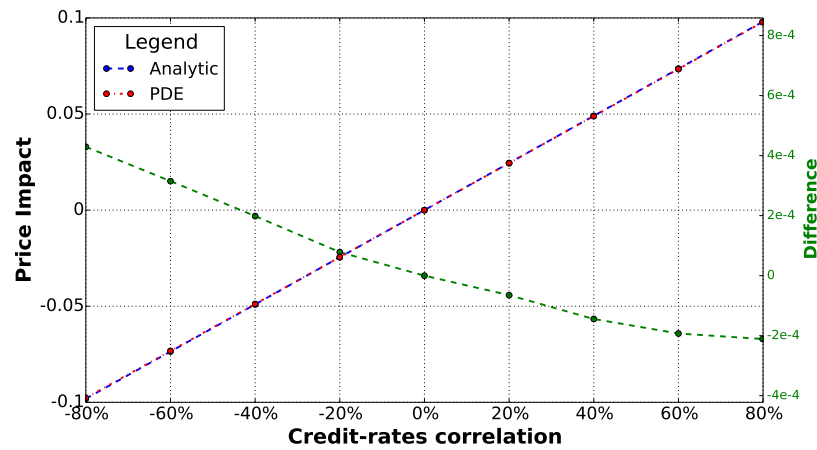


Figure 1. Impact of correlation on PV for a 5y maturity interest rate swap extinguisher.

It is from here a straightforward matter of differentiation to quantify the model uncertainty associated with the parameter ρ . For the coupon flows, there is no such dependency to leading order. For the Libor flows, we have

$$\frac{\partial PV_{Libor}^{(i)}}{\partial \rho} \sim -\frac{B(0, t_i) \Delta L^{(i)}}{\rho D(t_{i-1}, t_i)}. \quad (41)$$

and, again ignoring the higher order indirect impact of the λ_j through the (risky) discount factors, we obtain

$$\frac{\partial PV_{Libor}^{(i)}}{\partial \lambda_j} \approx -\frac{B(0, t_i)}{D(t_{i-1}, t_i)} \int_{T_{j-1}}^{(t_i \vee T_{j-1}) \wedge T_j} \phi_A^{(i)}(v) dv. \quad (42)$$

From (4), we infer that, if the uncertainty associated with ρ is $\Delta \rho$, the model uncertainty associated with an interest rate swap extinguisher calibrated to risky bond prices is

$$\text{Uncertainty} \approx \Delta \rho \left| \sum_{i=1}^N \frac{\partial PV_{Libor}^{(i)}}{\partial \rho} \right| \quad (43)$$

and, if calibration is to CDS rates:

$$\text{Uncertainty} \approx \Delta \rho \left| \sum_{i=1}^N \left(\sum_{j=1}^n \frac{\partial PV_{Libor}^{(i)}}{\partial \lambda_j} f'_j(\rho) + \frac{\partial PV_{Libor}^{(i)}}{\partial \rho} \right) \right|, \quad (44)$$

with $f'_j(\rho)$ given by (34). Notice that, in the latter case, the impact of calibration adjustment is such as to reduce the overall uncertainty (for either a payer or a receiver swap), so ignoring it would be to take a conservative approach.

5.2. Contingent CDS

We consider next a contingent CDS on an interest rate swap with 10 years to maturity, paying semi-annual Libor + 40 bp and receiving a quarterly fixed coupon of 250 bp. The result below is from Turfus (2017b). We look to calculate the cost of providing protection against default of the counterparty, in other words the counterparty value adjustment (CVA) associated with the payer swap position. The value of protection up to some horizon T will be governed by the nonhomogeneous version of the governing PDE:

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \bar{r}(t) - \bar{\lambda}(t) - \phi_\epsilon(x, y, t) \right) f(x, y, t) = -\lambda(y, t) P_{def}(x, t), \quad (45)$$

with $P_{def}(x, \tau)$ the protection payoff in the event of default at time τ , subject to the final condition $f(x, y, T) = 0$. For the swap defined above, we can write

$$P_{def}(x, \tau) = \max \left\{ (1 - R) \sum_{i=1}^N \left(V_L^{(i)}(x, \tau) - c \Delta_i V_F^{t_i}(x, \tau) \right), 0 \right\}, \quad (46)$$

with R the counterparty recovery rate, where the $V_L^{(i)}$ represent the PVs of the Libor flows and the $V_F^{t_i}$ those of the fixed coupon payments associated with the respective payment periods. Using a first order approximation based on (A18), we have

$$\begin{aligned} V_F^{t_i}(x, t) &= X^{t_i}(x, t) \\ &\sim D(t, t_i) (1 - x B^*(t_i - t)), \end{aligned} \quad (47)$$

with errors $= \mathcal{O}(\epsilon_r^2)$, where $B^*(\cdot)$ is given by (A12). Likewise, we have to the same level of accuracy

$$\begin{aligned} V_L^{(i)}(x, t) &= \begin{cases} X^{t_{i-1}}(x, t) - X^{t_i}(x, t), & t \leq t_{i-1}, \\ \frac{X^{t_i}(x, t)}{X^{t_i}(x, t_{i-1})} - X^{t_i}(x, t), & t_{i-1} < t < t_i. \end{cases} \\ &\sim \begin{cases} D(t, t_{i-1}) (1 - x B^*(t_{i-1} - t)) - D(t, t_i) (1 - x B^*(t_i - t)), & t \leq t_{i-1}, \\ D(t_{i-1}, t)^{-1} \frac{1 - x B^*(t_i - t)}{1 - x B^*(t_{i-1} - t)} - D(t, t_i) (1 - x B^*(t_i - t)), & t_{i-1} < t < t_i. \end{cases} \end{aligned} \quad (48)$$

The solution of (45) is by standard application of the Green's function expansion (A1): only the leading order term $G_{0,0}(\cdot)$ is needed for our purposes. We conclude following Turfus (2017b) that, with $\mathcal{O}(\epsilon_r^2 + \epsilon_\lambda^2)$ relative error, the cost of protection purchased at $t = 0$ on a payer swap is given by

$$V_{\text{protection}} \sim (1 - R) \sum_{i=1}^N \left(f_L^{(i)} - c \Delta_i f_F^{t_i} \right), \quad (49)$$

where

$$f_F^w := \int_0^{w \wedge T} \bar{\lambda}(v) B(0, v) D(v, w) N(-d_1(\xi^*(v), v)) dv, \quad (50)$$

$$\begin{aligned} f_L^{(i)} &:= \left(D(t_{i-1}, t_i)^{-1} - 1 \right) f_F^{t_i} + B^*(t_i - t_{i-1}) \int_0^{t_i \wedge T} \bar{\lambda}(v) B(0, v) D(v, t_i) \\ &\quad \left(\gamma(t_{i-1}, v) I_{r\lambda}(v \wedge t_{i-1}) N(-d_1(\xi^*(v), v)) + e^{-\alpha_r |v - t_{i-1}|} I_r(v \wedge t_{i-1}) \frac{N'(-d_1(\xi^*(v), v))}{\sqrt{I_r(v)}} \right) dv, \end{aligned} \quad (51)$$

with $I_r(\cdot)$ and $I_{r\lambda}(\cdot)$ given by (A7) and (A9), respectively, and

$$d_1(x, v) := \frac{x - I_{r\lambda}(v)}{\sqrt{I_r(v)}}, \quad (52)$$

$$\xi^*(v) := \inf \{x \mid P_{def}(x, v) > 0\}, \quad (53)$$

where the latter expression need only be calculated to leading order, whence $x_{t_{i-1}}$ can be replaced by x in (48). We further suppose formally that $P_{def}(\cdot) = \mathcal{O}(\epsilon_r)$ as $\epsilon_r \rightarrow 0$ so that the stochastic effects have a non-trivial impact, which is the situation of interest to us.

A comparison of (49) against the results of a Monte Carlo simulation is reproduced from Turfus (2017b) in Figure 2 to illustrate a typical parameter dependence structure and to indicate the level of accuracy that is furnished by our asymptotic method. The credit intensity in this case is 640 bp so not particularly “small”; the local volatility was taken to be 70% with a mean reversion rate of 0.3. The interest rate market was as in Section 5.1. The contract provided protection on the full value of the swap (assuming no recovery) for six years in return for semi-annual coupon payments of 400 bp.

The notional was again taken to be 100. As can be seen, the use of our linear approximation approach to the model risk remains good, with the discrepancy between the two modelling approaches unlikely to exceed a few basis points of notional.

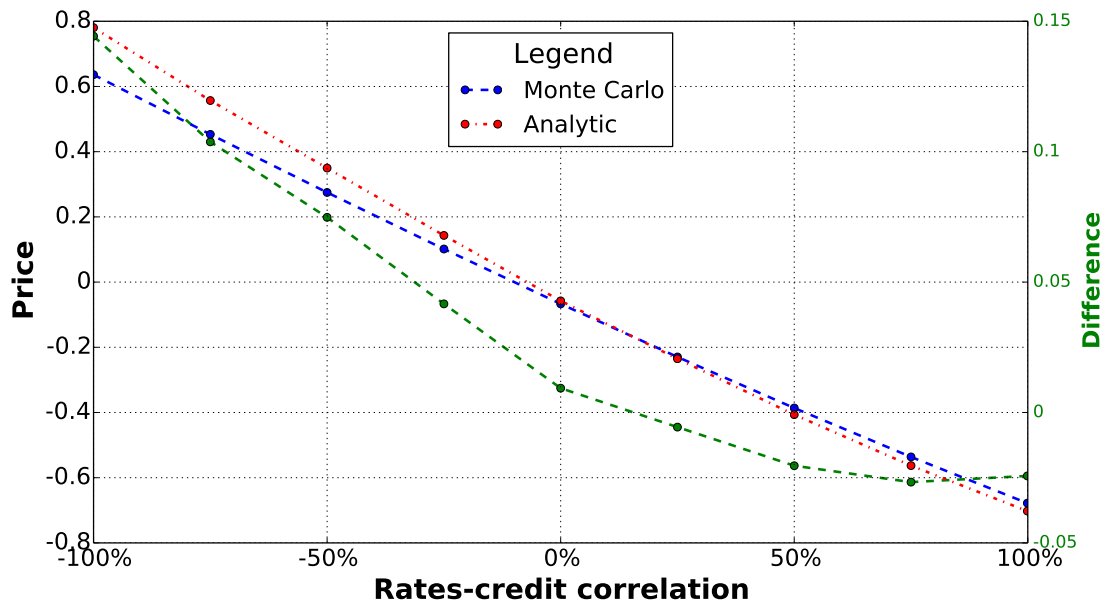


Figure 2. PV dependence of interest rate swap protection on rates–credit correlation level.

It is again a matter of straightforward differentiation to obtain an expression for the correlation risk associated with this modelling approach. To that end, we note that the relative impact through $d_1(\cdot)$ in the *second* term in (51) will be $\mathcal{O}(\epsilon_\lambda(\epsilon_r + \epsilon_\lambda))$, so we ignore this effect for our leading order calculation. We further note that, from (31), the relative impact of correlation through calibration is $\mathcal{O}(\epsilon_r\epsilon_\lambda)$ and furthermore likely to cancel between legs, so we ignore this too. We obtain

$$\begin{aligned} \text{Uncertainty} \approx \Delta\rho \left| \sum_{i=1}^N \int_0^{t_i \wedge T} \bar{\lambda}(v) B(0, v) D(v, t_i) \frac{\partial I_{r\lambda}(v \wedge t_{i-1})}{\partial \rho} \right. \\ \left. \left(\left(D(t_{i-1}, t_i)^{-1} - 1 - c\Delta_i \right) \frac{N'(-d_1(\xi^*(v), v))}{\sqrt{I_r(v)}} \right. \right. \\ \left. \left. + B^*(t_i - t_{i-1}) \gamma(t_{i-1}, v) N(-d_1(\xi^*(v), v)) \right) dv \right|. \quad (54) \end{aligned}$$

It may be suggested that the computational effort required here could become burdensome if N were large. However, the greatest computational effort will be involved in computing $\xi^*(v)$ and the associated cumulative normal. The values of the latter can be tabulated in advance for a range of $v \in [0, T]$ then interpolation used in the integration. Furthermore, for $T \equiv t_k$, we can re-express

$$\int_0^{t_i \wedge T} \equiv \sum_{j=1}^{i \wedge k} \int_{t_{j-1}}^{t_j}$$

and factor the integrand into the product of a v -dependent term and an i -dependent term, the latter of which can be taken outside the integral. This means we must integrate numerically from 0 to T only once, which is comparatively little effort. This approach was used to good effect by the author in the computations described in Turfus (2017b).

5.3. Capped Libor Flows

Finally, we consider the impact of capping a Libor flow such as was considered in Section 5.1 above at some level $K > 0$, in this case focusing only on the leading order contribution. The calculation is similar to that presented in Turfus (2017c) where the Libor payment was assumed to be in a foreign currency, but the results below are new and presented here for the first time. The payoff at time t_i for a payment period $[t_{i-1}, t_i]$ is given in previously defined notation by

$$\begin{aligned} \text{Payoff} &= \min\{X^{t_i}(x_{t_{i-1}}, t_{i-1})^{-1} - 1, K\Delta_i\} \\ &\sim \min\{D(t_{i-1}, t_i)^{-1}(1 + x_{t_{i-1}}B^*(t_i - t_{i-1})) - 1, K\Delta_i\} \end{aligned} \quad (55)$$

with errors $= \mathcal{O}(\epsilon_r^2)$. As in the previous section, we assume the payoff to be $\mathcal{O}(\epsilon_r)$ as $\epsilon_r \rightarrow 0$ so that the stochastic effects have a non-trivial impact in that limit. Because of the appearance of $x_{t_{i-1}}$ in the above expression, we must first compute the PV as of t_{i-1} . We obtain by straightforward application of our leading order Green's function $G_{0,0}$:

$$f(x_{t_{i-1}}, y_{t_{i-1}}, t_{i-1}) \sim B(t_{i-1}, t_i) \min\{D(t_{i-1}, t_i)^{-1}(1 + x_{t_{i-1}}B^*(t_i - t_{i-1})) - 1, K\Delta_i\}. \quad (56)$$

To proceed, we define the value

$$x^* := \frac{D(t_{i-1}, t_i)(1 + K\Delta_i) - 1}{B^*(t_i - t_{i-1})} \quad (57)$$

as the (asymptotic) representation of the value of $x_{t_{i-1}}$ at which the cap K is hit. Applying our (leading order) Green's function again to the payoff at t_{i-1} to obtain the PV at $t = 0$, we obtain

$$\begin{aligned} f(0, 0, 0) &\sim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{0,0}(0, 0, 0; \xi, \eta, t_{i-1}) f(\xi, \eta, t_{i-1}) d\xi d\eta \\ &\sim B(0, t_i)(D(t_{i-1}, t_i)^{-1} - 1) - \int_{-\infty}^{\infty} \int_{x^*}^{\infty} G_{0,0}(0, 0, 0; \xi, \eta, t_{i-1}) \\ &\quad \left(D(t_{i-1}, t_i)^{-1}(1 + \xi B^*(t_i - t_{i-1})) - 1 - K\Delta_i \right) d\xi d\eta. \end{aligned} \quad (58)$$

Carrying out the required integrations, we conclude

$$\begin{aligned} PV_{CappedLibor}^{(i)} &= f(0, 0, 0) \\ &\sim B(0, t_i) \left(\left(D(t_{i-1}, t_i)^{-1} - 1 \right) N(d_1(x^*, t_{i-1})) + K\Delta_i N(-d_1(x^*, t_{i-1})) \right. \\ &\quad \left. - D(t_{i-1}, t_i)^{-1} B^*(t_i - t_{i-1}) \sqrt{I_r(t_{i-1})} N'(-d_1(x^*, t_{i-1})) \right) \end{aligned} \quad (59)$$

with $\mathcal{O}(\epsilon_r + \epsilon_\lambda)$ relative error. In a similar vein, for a Libor flow *floored* at K , we have

$$\begin{aligned} PV_{FlooredLibor}^{(i)} &\sim B(0, t_i) \left(K\Delta_i N(d_1(x^*, t_{i-1})) + \left(D(t_{i-1}, t_i)^{-1} - 1 \right) N(-d_1(x^*, t_{i-1})) \right. \\ &\quad \left. + D(t_{i-1}, t_i)^{-1} B^*(t_i - t_{i-1}) \sqrt{I_r(t_{i-1})} N'(-d_1(x^*, t_{i-1})) \right). \end{aligned} \quad (60)$$

On this occasion, none of the terms involving $d_1(\cdot)$ should be neglected in calculating the model uncertainty since they constitute the leading order impact of correlation. They furthermore impact only one leg, not both, so there will be no cancellation between legs as in the previous case. Carrying out the necessary differentiation, we obtain, for both the capped and the floored cases:

$$\text{Uncertainty} \sim \Delta \rho \left| B(0, t_i) \frac{\partial I_{r\lambda}(t_{i-1})}{\partial \rho} \frac{N'(-d_1(x^*, t_{i-1}))}{\sqrt{I_r(t_{i-1})}} \left(D(t_{i-1}, t_i)^{-1} - 1 - K\Delta_i \right. \right. \\ \left. \left. - D(t_{i-1}, t_i)^{-1} B^*(t_i - t_{i-1}) \sqrt{I_r(t_{i-1})} d_1(x^*, t_{i-1}) \right) \right|. \quad (61)$$

Again, we ignore the second order impact of correlation through calibration. We mention for completeness that, if the cap or floor is strongly in or out of the money, the uncertainty resulting from (61) will be small and possibly comparable with that of the plain Libor. We may in that case look to compute the latter separately using (44).

6. Conclusions

We have proposed a framework for the quantification of model risk in credit derivatives pricing in circumstances where the correlation between rates and credit is either uncertain in its value or not included in the calculation. We considered in particular the cases of (a) an interest rate swap extinguisher, (b) a contingent CDS on an interest rate swap underlying, and (c) an extinguisher with capped or floored Libor flows. We derived explicit analytic expressions for the model risk as a function of the degree of uncertainty associated with the correlation, under an asymptotic assumption of the interest rate and the credit default intensity being small and taking into account the potential impact of correlation on model calibration. We propose that the results obtained are accurate enough for risk management purposes.

Although the cases considered here involve rather simple modelling considerations, we suggest that the approach advocated has much wider application, including for parameters other than rates–credit correlation in other multi-factor models. In particular, it is possible to look at asymptotic modelling involving also the price of a spot underlying such as an equity, an FX rate or an inflation rate. These quantities could further be assumed to jump in value contingent on default. Modelling then requires a three-dimensional diffusion process (possibly four, since two interest rates may appear, either or both of which may be assumed stochastic). Pricing of defaultable FX swaps, quanto CDS, contingent CDS on FX or equity options, convertible bonds and contingent convertible (CoCo) bonds can all be handled and analytic expressions for model uncertainty obtained in the manner specified above. For some examples of asymptotic formulae amenable to such analysis, see [Turfus and Schubert \(2017\)](#), [Turfus \(2017c\)](#), [Turfus \(2017d\)](#), [Turfus \(2017e\)](#) and [Turfus \(2018\)](#). In cases where a possible jump at default is assumed, the model uncertainty associated with uncertainty in the expected jump size can also easily be obtained as an analytic expression.

Much work has also been done by other authors using perturbation approaches to obtain analytic approximations for prices of numerous option types under local and/or stochastic volatility modelling assumptions. See, for example, [Pagliarani and Pascucci \(2011\)](#), who considered equity option pricing under a local volatility assumption, obtaining a perturbation expansion for the relevant Green's function much as we did here, and using it to derive asymptotic expressions for option prices. Their approach was applied also to Asian option pricing in [Foschi et al. \(2013\)](#) and extended, with the use of some Fourier analysis, to incorporate Lévy jumps in the dynamics of the spot underlying in [Pagliarani and Pascucci \(2013\)](#). A review of a number of other papers which have presented asymptotic pricing formulae in recent years has been given by [Turfus and Schubert \(2017\)](#). Our model uncertainty methodology is potentially applicable also to the results of such work. An interesting prospect for future work would be to combine asymptotic modelling of stochastic rates and local-stochastic volatility, as was done by [Funahashi \(2015\)](#), and look at the resultant model uncertainty in options pricing.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Green's Function

From the analysis of [Turfus \(2017a\)](#), we infer that the Green's function solution of (25) can be expanded as

$$G(x, y, t; \xi, \eta, v) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} G_{i,j}(x, y, t; \xi, \eta, v), \quad (\text{A1})$$

with $G_{i,j}(\cdot) = \mathcal{O}(\epsilon_r^i \epsilon_\lambda^j)$. We will for the present purposes be interested only in terms up to second order, with consequent $\mathcal{O}(\epsilon_r^3 + \epsilon_\lambda^3)$ errors in $G(\cdot)$. We will in all cases be interested in “free-boundary” Green's function solutions which tend to zero as $x, y \rightarrow \pm\infty$. The leading order Green's function solution subject to these conditions is straightforwardly deduced. It is given by:

$$G_{0,0}(x, y, t; \xi, \eta, v) = B(t, v) \frac{\partial^2}{\partial \xi \partial \eta} N_2(\xi - x e^{-\alpha_r(v-t)}, \eta - y e^{-\alpha_\lambda(v-t)}; R(t, v)), \quad t < v, \quad (\text{A2})$$

where $N_2(x, y; R(t, v))$ is a bivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t, v) := \begin{pmatrix} I_r(t, v) & I_{r\lambda}(t, v) \\ I_{r\lambda}(t, v) & I_\lambda(t, v) \end{pmatrix} \quad (\text{A3})$$

with

$$I_r(t_1, t_2) := \int_{t_1}^{t_2} e^{-2\alpha_r(t_2-u)} \sigma_r^2(u) du, \quad (\text{A4})$$

$$I_\lambda(t_1, t_2) := \int_{t_1}^{t_2} e^{-2\alpha_\lambda(t_2-u)} \sigma_\lambda^2(u) du, \quad (\text{A5})$$

$$I_{r\lambda}(t_1, t_2) := \rho \int_{t_1}^{t_2} e^{-(\alpha_r + \alpha_\lambda)(t_2-u)} \sigma_r(u) \sigma_\lambda(u) du. \quad (\text{A6})$$

For future notational convenience, we also define

$$I_r(t) := I_r(0, t), \quad (\text{A7})$$

$$I_\lambda(t) := I_\lambda(0, t), \quad (\text{A8})$$

$$I_{r\lambda}(t) := I_{r\lambda}(0, t). \quad (\text{A9})$$

Following [Turfus \(2017a\)](#), we deduce at first order:

$$G_{1,0}(x, y, t; \xi, \eta, v) = - \left(x B^*(v-t) + \frac{I^*(t, v)}{e^{-\alpha_r(v-t)}} \frac{\partial}{\partial x} \right) G_{0,0}(x, y, t; \xi, \eta, v) \quad (\text{A10})$$

and

$$G_{0,1}(x, y, t; \xi, \eta, v) = - \int_t^v \bar{\lambda}(t_1) \left(\mathcal{E}_y(e^{-\alpha_\lambda(t_1-t)} y_t) \mathcal{M}_{t,t_1} - 1 \right) G_{0,0}(x, y, t; \xi, \eta, v) dt_1, \quad (\text{A11})$$

where we have defined

$$B^*(\tau) := \frac{1 - e^{-\alpha_r \tau}}{\alpha_r}, \quad (\text{A12})$$

$$I^*(t, v) := \int_t^v e^{-\alpha_r(v-u)} I_r(t, u) du, \quad (\text{A13})$$

$$\mathcal{M}_{t_1, t_2} G_{0,0}(x, y, t; \xi, \eta, v) := G_{0,0}(x, y, t; \xi, \eta - e^{-\alpha_\lambda(v-t_2)} I_\lambda(t_1, t_2), v). \quad (\text{A14})$$

Proceeding to second order, we obtain

$$G_{2,0}(x, y, t; \xi, \eta, v) = \int_t^v \left(x e^{-\alpha_r(t_1-t)} + \frac{I_r(t, t_1)}{e^{-\alpha_r(t_1-t)}} \frac{\partial}{\partial x} \right) \int_{t_1}^v \left(x e^{-\alpha_r(t_2-t)} + \frac{I_r(t_1, t_2)}{e^{-\alpha_r(t_2-t)}} \frac{\partial}{\partial x} \right) G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ + \int_t^v (I^*(t, t_1) - \gamma_{2,0}^*(t_1)) dt_1 G_{0,0}(x, y, t; \xi, \eta, v), \quad (A15)$$

$$G_{0,2}(x, y, t; \xi, \eta, v) = \int_t^v \bar{\lambda}(t_1) \left(\mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) \mathcal{M}_{t,t_1} - 1 \right) \int_{t_1}^v \bar{\lambda}(t_2) \left(\mathcal{E}_y \left(e^{-\alpha_\lambda(t_2-t)} y_t \right) \mathcal{M}_{t_1,t_2} - 1 \right) G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ + \int_t^v \bar{\lambda}(t_1) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) \int_{t_1}^v \bar{\lambda}(t_2) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_2-t)} y_t \right) \left(\exp \left(e^{-\alpha_\lambda(t_2-t_1)} I_\lambda(t, t_1) \right) - 1 \right) \mathcal{M}_{t,t_1} \mathcal{M}_{t_1,t_2} G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ - \int_t^v \gamma_{0,2}^*(t_1) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) \mathcal{M}_{t,t_1} G_{0,0}(x, y, t; \xi, \eta, v) dt_1 \quad (A16)$$

and

$$G_{1,1}(x, y, t; \xi, \eta, v) = \int_t^v \left(x e^{-\alpha_r(t_1-t)} + \frac{I_r(t, t_1)}{e^{-\alpha_r(t_1-t)}} \frac{\partial}{\partial x} \right) \int_{t_1}^v \bar{\lambda}(t_2) \left(\mathcal{E}_y \left(e^{-\alpha_\lambda(t_2-t)} y_t \right) \mathcal{M}_{t_1,t_2} - 1 \right) G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ + \int_t^v \bar{\lambda}(t_1) \left(\mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) \mathcal{M}_{t,t_1} - 1 \right) \int_{t_1}^v \left(x e^{-\alpha_r(t_2-t)} + \frac{I_r(t_1, t_2)}{e^{-\alpha_r(t_2-t)}} \frac{\partial}{\partial x} \right) G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ + \int_t^v \int_{t_1}^v \bar{\lambda}(t_2) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_2-t)} y_t \right) e^{-\alpha_\lambda(t_2-t_1)} I_{r\lambda}(t, t_1) \mathcal{M}_{t_1,t_2} G_{0,0}(x, y, t; \xi, \eta, v) dt_2 dt_1 \\ + \int_t^v \bar{\lambda}(t_1) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) B^*(v - t_1) I_{r\lambda}(t, t_1) \mathcal{M}_{t,t_1} G_{0,0}(x, y, t; \xi, \eta, v) dt_1 \\ - \int_t^v \gamma_{1,1}^*(t_1) \mathcal{E}_y \left(e^{-\alpha_\lambda(t_1-t)} y_t \right) \mathcal{M}_{t,t_1} G_{0,0}(x, y, t; \xi, \eta, v) dt_1. \quad (A17)$$

Note that the order of integration between t_1 and t_2 has been reversed compared to [Turfus \(2017a\)](#). It is a straightforward application of Fubini's theorem to derive the alternative expressions from the above.

Finally, determination of the unknown $\gamma_{ij}^*(\cdot)$ functions is achieved by calibration of our model consistent with the no-arbitrage conditions (12) and (13). We must consider the consistent pricing in the former case of a risk-free cash flow, and in the latter case of a risky cash flow, as we now show.

Appendix A.1. Calibration

Appendix A.1.1. Pricing of Risk-Free Cash Flow

The calculation for a risk-free cash flow in our model is very similar to that performed by [Horvath et al. \(2017\)](#) and essentially corresponds to taking the distinguished limit as $\epsilon_\lambda \rightarrow 0$ then $\epsilon_r \rightarrow 0$. The same result is naturally obtained, namely that $f_t^T = X^T(x, t)$, where, with the convention that $F_{ij}(x, t) = \mathcal{O}(\epsilon_r^i \epsilon_\lambda^j)$,

$$X^T(x, t) \sim D(t, T) (1 - F_{1,0}(x, t) + F_{2,0}(x, t)) \quad (A18)$$

with $\mathcal{O}(\epsilon_r^3)$ errors, and our Green's function gives rise to

$$F_{1,0}(x, t) = xB^*(T - t), \quad (\text{A19})$$

$$F_{2,0}(x, t) = \frac{1}{2}x^2B^*(T - t)^2 + \int_t^T (I^*(t, v) - \gamma_{2,0}^*(v))dv. \quad (\text{A20})$$

Of interest to us here is the conclusion that, setting $x = y = t = 0$ in (A18), satisfying (12) above to second order accuracy requires us to choose

$$\gamma_{2,0}^*(t) = I^*(0, t), \quad (\text{A21})$$

which is $\mathcal{O}(\epsilon_r^2)$, whence, on carrying out the required integration and using (A4), we can re-express

$$F_{2,0}(x, t) = \frac{1}{2}(x^2 - I_r(t))B^*(T - t)^2 - I^*(0, t)B^*(T - t). \quad (\text{A22})$$

The second term here is the convexity correction associated with the chosen money market numéraire, which term noticeably vanishes both at $t = 0$ and at $t = T$ when the PV is known deterministically.

Appendix A.1.2. Pricing of Risky Cash Flow

We continue by writing the price at time t of a risky (zero recovery) cash flow at time T as $f_t^T = Y^T(x_t, y_t, t)$, noting that, in this case, $P(x) = 1$ and $f_0^T = Y^T(0, 0, 0) = B(0, T)$. We look to derive the general functional form of $Y^T(\cdot)$ implied by our model, and in the process to determine the conditions on $\lambda^*(t)$ necessary to satisfy (13) above. Applying our second order Green's function to this problem, we conclude

$$Y^T(x, y, t) \sim B(t, T) (1 - F_{1,0}(x, t) - F_{0,1}(y, t) + F_{2,0}(x, t) + F_{1,1}(x, y, t) + F_{0,2}(y, t)) \quad (\text{A23})$$

with $\mathcal{O}(\epsilon_r^3 + \epsilon_\lambda^3)$ error, where the $F_{i,0}(x, t)$ are as defined above for $i = 1, 2$ and we deduce, following Turfus (2017a),

$$F_{0,1}(y, t) = \int_t^T \bar{\lambda}(v) \left(\mathcal{E}_y \left(e^{-\alpha_\lambda(v-t)} y_t \right) - 1 \right) dv, \quad (\text{A24})$$

$$F_{0,2}(y, t) = \frac{1}{2}F_{0,1}^2(y, t) - \int_t^T \gamma_{0,2}^*(v) \mathcal{E}_y \left(e^{-\alpha_\lambda(v-t)} y_t \right) dv \\ + \int_t^T \bar{\lambda}(v) \mathcal{E}_y \left(e^{-\alpha_\lambda(v-t)} y_t \right) \int_t^v \bar{\lambda}(u) \mathcal{E}_y \left(e^{-\alpha_\lambda(u-t)} y_t \right) \left(\exp \left(e^{-\alpha_\lambda(v-u)} I_\lambda(u) \right) - 1 \right) du dv, \quad (\text{A25})$$

$$F_{1,1}(x, y, t) = xB^*(T - t)F_{0,1}(y, t) - \int_t^T \gamma_{1,1}^*(v) \mathcal{E}_y \left(e^{-\alpha_\lambda(v-t)} y_t \right) dv \\ + \int_t^T \bar{\lambda}(v) \mathcal{E}_y \left(e^{-\alpha_\lambda(v-t)} y_t \right) \int_t^v e^{-\alpha_\lambda(v-u)} I_{r\lambda}(u) du dv \\ + \int_t^T \int_t^v \bar{\lambda}(u) \mathcal{E}_y \left(e^{-\alpha_\lambda(u-t)} y_t \right) e^{-\alpha_r(v-u)} I_{r\lambda}(u) du dv. \quad (\text{A26})$$

Setting $x = y = t = 0$, we find that the no-arbitrage condition $Y^T(0, 0, 0) = B(0, T)$ is satisfied by the expression in (A23) to second order accuracy iff we choose

$$\gamma_{0,2}^*(t) = \bar{\lambda}(t) \int_0^t \bar{\lambda}(u) \left(\exp \left(e^{-\alpha_\lambda(t-u)} I_\lambda(u) \right) - 1 \right) du, \quad (\text{A27})$$

$$\gamma_{1,1}^*(t) = \int_0^t \left(\bar{\lambda}(t) e^{-\alpha_\lambda(t-u)} + \bar{\lambda}(u) e^{-\alpha_r(t-u)} \right) I_{r\lambda}(u) du, \quad (\text{A28})$$

which are $\mathcal{O}(\epsilon_\lambda^2)$ and $\mathcal{O}(\epsilon_r \epsilon_\lambda)$, respectively. This completes the calibration of our model to second order.

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